Presentations for Topological Modalities

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  - We have a universe $\mathcal{U}$ and a subtype $\mathrm{Prop}_{\mathcal{U}} \subseteq \mathcal{U}$.
  - Given a function $B : A \to \mathcal{U}$, we can form a type $\sum_{a:A} B(a) : \mathcal{U}$.

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- HoTT = internal language of $(\infty, 1)$-topoi.[1]

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Goal

Work with subtopoi in HoTT in a sheaf theoretic way.
Use this in different synthetic maths.

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- Internally a map is surjective iff all its fibers are inhabitted.

Subtopoi in HoTT

Consider a family of propositions $P : I \to \mathrm{Prop}_{\mathcal{U}}$

Definition ([3])

A type X is a **sheaf** for P if for all $i : I$ the natural map

$$X \to (P(i) \to X)$$

is an equivalence. We define $\mathcal{U}_P := \{X : \mathcal{U} \mid X \text{ is a sheaf}\}$.

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Definition ([3])

The choice of a subuniverse and sheafification functor, such that there exists a family of propositions generating it is called a **topological modality**.

[3] Spitters, Shulman, and Rijke 2020.

Presentations

Definition ([4])

A **presentation of a topological modality** is a collection T of types closed under Σ, containing 1.
The topological modality **presented by** T is given by the propositions $\|X\|$ for X in T.

Note: $\|X\|$ is the propositional truncation of X, or the image factorisation of $X \to 1$.

[4] Moeneclaey 2024.

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More interesting examples will need new axioms in HoTT...

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A **cover** for a presentation T is a map $f : X \to Y$ such that for all $y : Y$, the fiber $f^{-1}(y)$ is in T.

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Lemma

- *Any equivalence is a cover.*
- *Covers are closed under pullback and comoposition.*

Sheaf Conditions

- HoTT is higher:

Set / 0-type	Sheaf of sets
1-type	Sheaf of groupoids
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- So expect to get higher sheaf conditions.
- For each n, want a condition for an n-type to be a sheaf.
- Need to use homotopy constructs.

Sheaf Conditions

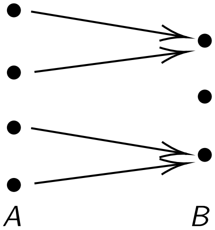
Definition

Given $f : A \to X$ and $g : B \to X$ their **join** is the pushout

$$\begin{array}{ccc} A \times_X B & \longrightarrow & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & A *_X B \end{array}$$

Given $f : A \to X$ write A_X^{*n} for the n-fold iterated join of f with itself.

Joins



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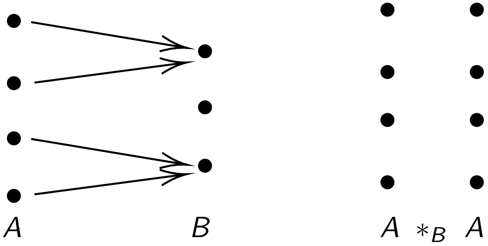

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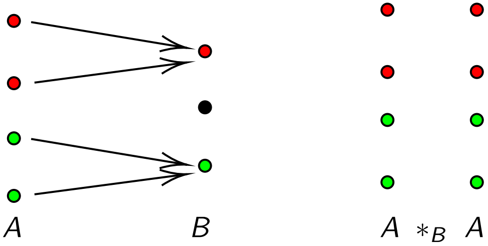

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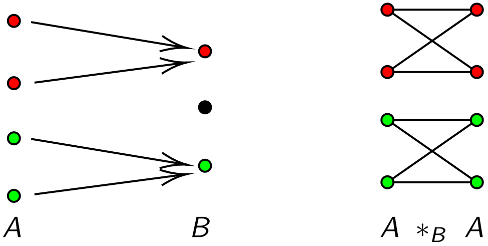

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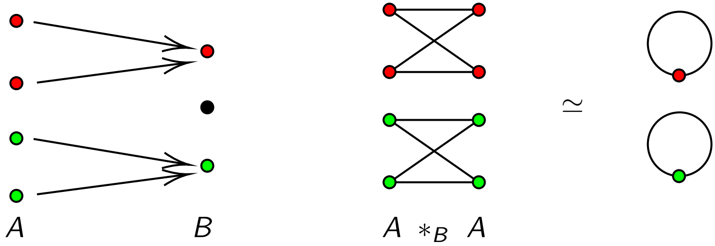

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Theorem ([5])

For any $f : A \to B$ we have

$$\mathrm{colim}(A \to A_B^{*} \to A_B^{*2} \to A_B^{*3} \to \cdots) \simeq \mathrm{im}\, f$$

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Lemma

Let $A : \mathcal{U}$ and $X : \mathcal{U}$ be an n-type. The map $A^{*(n+2)} \to A^{*(n+3)}$ induces an equivalence $(A^{*(n+3)} \to X) \simeq (A^{*(n+2)} \to X)$.

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Surjectivity on an n-type is controlled by $(n+2)$-fold joins.

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Sheaf Conditions

Fix a presentation T.

Theorem (Sheaf Condition)

Let X be an n-type. Then X is a sheaf for T iff for all T-covers $f : A \to B$ the natural map

$$(B \to X) \to (A_B^{*n+2} \to X)$$

is an equivalence.

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Question: Why is this a sheaf condition?

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A 0-type X is a sheaf for T iff for all T-covers $f : A \to B$ the natural map

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Proof.

By squinting:

$$X(B) \to X(A) \rightrightarrows X(A \times_B A)$$

□

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Slight Inconvenience

We can only do this for each external natural number. There is no way to prove internally for all $n : \mathbb{N}$ that this holds.

Applications

- Let $\mathbb{T}$ be an algebraic theory.

[6] Cherubini, Coquand, and Hutzler 2023.
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- We axiomatise the classifying topos for $\mathbb{T}$.

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- If $\mathbb{T}$ is the theory of rings, then get "presheaf synthetic algebraic geometry"[6]
- If $\mathbb{T}$ is the theory of bounded distributive lattices with 0 and 1, get synthetic higher category theory.[7]

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Thank you!

References

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