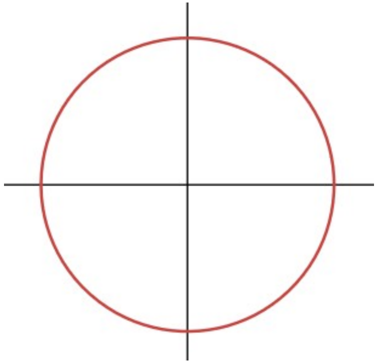
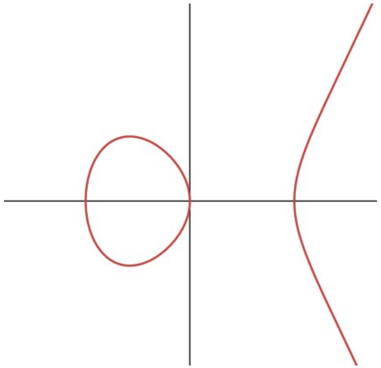
# Synthetic Algebraic Geometry

# What is Algebraic Geometry?

Study of solutions to polynomial equations:



$x^2 + y^2 = 1$



$y^2 = x^3 - x$

Sometimes over other rings!

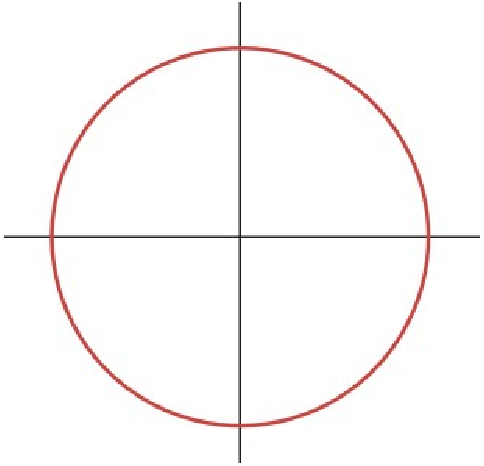
$x^n + y^n = z^n$ where $x, y, z \in \mathbb{Z}$

Sometimes not in rings (but things made out of them)!

Projective line: $\mathbb{P}^1$

# What is Algebraic Geometry?

Study of solutions to polynomial equations... **using algebraic objects**



$$x^2 + y^2 = 1$$

$$\text{Polynomial functions } X \to \mathbb{C}$$

$$\cong$$

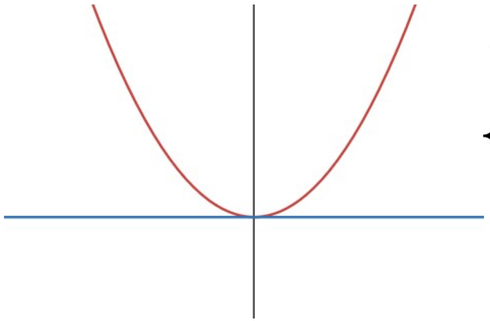
$$\mathbb{C}[x,y]/(x^2+y^2-1)$$

# Classical vs Modern

Classical algebraic geometry studies actual solution sets.

$$\{\bar{x} \in R^n \mid p_1(\bar{x}) = \cdots = p_m(\bar{x}) = 0\}$$

But for a number of reasons these aren't good enough:



$$\{x, y \in \mathbb{C}^2 \mid y = x\} \cap \{x, y \in \mathbb{C}^2 \mid y = 0\} = \{(0,0)\}$$
$$\{x, y \in \mathbb{C}^2 \mid y = x^2\} \cap \{x, y \in \mathbb{C}^2 \mid y = 0\} = \{(0,0)\}$$

Where is the double root??

# Classical vs Modern

Classical methods only work over nice rings (= algebraically closed fields)
But there *should* be a method to work geometrically over weird rings.

E.g: **Weil Conjectures** (1949) (= Riemann Hypothesis over finite fields)

Given a nice algebraic set $X$ over a finite field $\mathbb{F}_q$ we can define a zeta function

$$\zeta(X, s) := \exp\left(\sum_{n=1}^{\infty} |X(\mathbb{F}_{q^n})| \frac{s^n}{n}\right)$$

1. ζ is rational
2. ζ satisfies Poincaré duality functional equation
3. Riemann Hypothesis
4. Relationship between ζ and Betti numbers

# Classical vs Modern

Weil conjectures suggest actually suggest the proof!

1. Give algebraic spaces a good notion cohomology theory
2. Prove the Lefschetz fixpoint theorem and Poincaré duality for this cohomology theory.
3. You have proved the Weil conjectures!

Easier said than done...

# The Grothendieck school

Idea:

- Flip the definition of solution sets
- Make it so that each ring $R$ has a “solution set” $\mathrm{Spec}(R)$
- $R$ is the ring of “algebraic functions” on $\mathrm{Spec}(R)$
- Make more complicated spaces by gluing spectrums together.

Then:

- Intersections are fixed!

$$\mathrm{Spec}(R[x,y]/(y-x^2)) \cap \mathrm{Spec}(R[x,y]/(y)]) = \mathrm{Spec}(R[x]/(x^2))$$

- It becomes easier to see how to define cohomology.

# The Grothendieck school

*The very idea of scheme is of infantile simplicity — so simple, so humble, that no one before me thought of stooping so low. So childish, in short, that for years, despite all the evidence, for many of my erudite colleagues, it was really “not serious”!*
- A. Grothendieck, translated by C. McLarty

Let’s see this childishly simple work

A **ringed space** is a pair $(X, \mathscr{O}_X)$ where $X$ is a topological space and $\mathscr{O}_X$ is a sheaf of rings on $X$. That is $\mathscr{O}_X : \mathcal{O}(X)^{\mathrm{op}} \to \mathrm{Set}$ satisfying a sheaf condition.

A **locally ringed space** is a ringed space $(X, \mathscr{O}_X)$ so that for all $x \in X$ the stalk $\mathscr{O}_{X,x} := \mathrm{colim}_{x \in U \subseteq X} \mathscr{O}(U)$ is a local ring.

Given a ring $R$ there is a locally ringed space $(\mathrm{Spec}(R), \mathscr{O}_{\mathrm{Spec}(R)})$ where

- $\mathrm{Spec}(R) = \{\mathfrak{p} \subset R \mid \mathfrak{p} \text{ is a prime ideal}\}$ endowed with the Zariski topology, generated by $D(f) := \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$

- The sheaf is defined on the basis by $\mathscr{O}_{\mathrm{Spec}(R)}(D(f)) = R[\frac{1}{f}]$

A locally ringed space is an **affine scheme** if it is isomorphic (as locally ringed spaces) to $(\mathrm{Spec}(R), \mathscr{O}_{\mathrm{Spec}(R)})$

A **scheme** is a locally ringed space $(X, \mathscr{O}_X)$ such that for each $x \in X$ there is an open neighbourhoud $x \in U \subset X$ so that $(U, \mathscr{O}_X|_U)$ is an affine scheme.

# The Grothendieck school

*There is no serious historical question of how Grothendieck found his definition of schemes. It was in the air. Serre has well said that no one invented schemes... The question is, what made Grothendieck believe he should use this definition to simplify an 80 page paper by Serre into some 1000 pages of Élements de Géometrie Algébrique?*
- C. McLarty

# Modern(ish) Methods

That was … complicated. But the idea is simple!

Upshot: We embed the (opposite) category of rings into somewhere we can glue more easily!

**Theorem** *The category* $\mathrm{Ring}^{\mathrm{op}}$ *is equivalent* $\mathrm{Aff}$ *of affine schemes.*

A category theorists way to do this: Embed into the free cocompletion!

$$\mathrm{Ring}^{\mathrm{op}} \hookrightarrow \mathrm{Fun}(\mathrm{Ring}, \mathrm{Set})$$

But don’t take all functors, only take ones that are correctly glued, and respect “open subsets”.

Take a category of **sheaves**. This is called the Zariski topos!

# Modern(ish) Methods

This is (sort of) why topoi were defined! Logic came later.

The Zariski topos, and a related topos (Etale topos) helped solve the Weil conjectures!

- Dwork proved the first Weil conjecture in 1960
- Grothendieck proved 1, 2, 4 of the Weil conjectures in 1965
- Deligne proved 3. (the Riemann hypothesis) in 1974

# Modern methods

Since the invention of topoi, they have become a tool to translate between logic and geometry.

To each topos we can give a model of (higher order) logic / type theory.
Given a formula $\varphi$ we can ask whether a given topos $\mathcal{E}$ believes this statement

**Question:** Is the internal logic of the Zariski topos a nice logic to do algebraic geometry?

# Internal logic of the Zariski topos

Small aspects of the internal logic have been studied by Anders Kock, Gavin Wraith, Myles Tierney...

The definitive work was done by Ingo Blechschmidt in his thesis (2017)

**Answer:** Not only is the internal logic useful, it can be used to study algebraic geometry itself entirely synthetically!

Ingo begun giving a set of *axioms* you can add to (intuitionistic) set theory / type theory.

This program has been continued and expanded by Felix Cherubini, Thiery Coquand, Matthias Hutzler, Hugo Moeneclaey, David Wärn...

**Surprise**: Moving from sets to higher types is really what made the internal logic useful!

# External vs Internal

| External | Internal |
|---|---|
| Forgetful $U$ : Ring $\to$ Set | Local ring $R$ |
| $A : Ring$ | $R$-algebra $\tilde{A}$ over $R$ |
| Affine scheme Spec$(A)$ | Internal Spec$(\tilde{A}) := \mathrm{Hom}_R(A, R)$ |
| $\vdots$ | $\vdots$ |

**Goal for this talk:** Define the internal notion of a scheme!